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# LETTERS TO THE EDITOR

## A NEW PERTURBATION TECHNIQUE WHICH IS ALSO VALID FOR LARGE PARAMETERS

JI-HUAN HE

*Shanghai University, Shanghai Institute of Applied Mathematics and Mechanics, Shanghai* 200072, *People's Republic of China*

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### 1. INTRODUCTION

There are many difficulties encountered in the application of perturbation techniques to the study of strong non-linear problems. Of these, one of the most frustrating is the fact that all classical perturbation techniques strongly rely on the assumption of the small parameter. To overcome the limitations, many novel techniques have been proposed in recent years. For example, Cheung *et al.* [1] propose a modified Lindstedt–Poincare method, and He [2, 3] proposes a homotopy perturbation technique. In the present paper, we will propose a new perturbation technique, which is valid not only for small parameters, but also for very large values of parameters.

### 2. BASIC IDEAS OF THE NEW METHOD

To illustrate the basic idea of the present note, we consider the well-known Duffing equation [4, 5]

$$u'' + u + \varepsilon u^3 = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{1}$$

For small values of $\varepsilon$, the classical perturbation methods are looking for a solution of equation (1) having the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{2}$$

a power series in $\varepsilon$ with coefficients that are independent of $\varepsilon$. Clearly, the leading term $u_0$ is the solution of the linear approximation

$$u'' + u = 0, \qquad u(0) = A, \qquad u'(0) = 0 \tag{3}$$

by setting $\varepsilon = 0$. Remember that in equation (2) the second term $\varepsilon u_1$ is a correction to the leading term $u_0$, and so on. Due to the fact $u_0$ is obtained upon setting $\varepsilon = 0$, which means that $u_0$ can be considered as an approximated solution to the original equation (1) only when $\varepsilon \ll 1$, the approximations obtained by perturbation methods are valid only for the case $\varepsilon \ll 1$.

If, however, the leading term $u_0$ in equation (2) is an approximate solution of equation (1) regardless of the values of the parameter $\varepsilon$, the obtained results are valid regardless of the parameters. But how to search for such a leading term $u_0$?

Supposing that the angular frequency of the system is $\beta$, which is unknown to be further determined, we can obtain the linearized Duffing equation, which reads

$$u'' + \beta^2 u = 0. \tag{4}$$

In view of equation (4), we can rewrite equation (1) in the form

$$u'' + \beta^2 u + \varepsilon(u^3 + \eta u) = 0, \qquad u(0) = A, \qquad u'(0) = 0, \tag{5}$$

where

$$\beta^2 + \varepsilon\eta = 1, \tag{6}$$

where $\eta$ is an unknown constant.

Supposing that the solution of equation (5) can also be expressed in the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{7}$$

where the parameter $\varepsilon$ need not be small in the present study.

Substituting equation (7) into equation (5) and equating coefficients of like powers of $\varepsilon$ yield the following equations:

$$u_0'' + \beta^2 u_0 = 0, \qquad u_0(0) = A, \qquad u_0'(0) = 0, \tag{8}$$

$$u_1'' + \beta^2 u_1 + u_0^3 + \eta u_0 = 0, \qquad u_1(0) = 0, \qquad u_1'(0) = 0. \tag{9}$$

Solving equation (8) results in

$$u_0 = A \cos \beta t. \tag{10}$$

Equation (9), therefore, can be rewritten as

$$u_1'' + \beta^2 u_1 + (\tfrac{3}{4} A^2 + \eta) A \cos \beta t + \tfrac{1}{4} A^3 \cos 3\beta t = 0. \tag{11}$$

Avoiding the presence of a secular term needs

$$\eta = -\tfrac{3}{4} A^2. \tag{12}$$

Solving equation (11), we obtain

$$u_1 = -\frac{A^3}{32\beta^2} (\cos \beta t - \cos 3\beta t). \tag{13}$$

If, for example, its first order approximation is sufficient, then we have

$$u = A \cos \beta t - \frac{\varepsilon A^3}{32\beta^2} (\cos \beta t - \cos 3\beta t), \tag{14}$$

where the angular frequency can be written in the form

$$\beta = \sqrt{1 - \varepsilon\eta} = \sqrt{1 + \tfrac{3}{4} \varepsilon A^2}. \tag{15}$$

Observe that for small $\varepsilon$, i.e., $0 < \varepsilon \ll 1$, it follows that

$$\beta = 1 + \tfrac{3}{8}\,\varepsilon A^2. \tag{16}$$

Consequently, in this limit, the present method gives exactly the same results as the standard Lindstedt–Poincare method [4, 5]. To illustrate the remarkable accuracy of the result obtained, we compare the approximate period

$$T = \frac{2\pi}{\sqrt{1 + 3\varepsilon A^2/4}} \tag{17}$$

with the exact one [2]

$$T_{ex} = \frac{4}{\sqrt{1+\varepsilon A^2}} \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - k\sin^2 x}} \quad \text{with } k = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)}. \tag{18}$$

What is rather surprising about the remarkable range of validity of equation (17) is that the actual asymptotic period $\varepsilon \to \infty$ also has high accuracy.

$$\lim_{\varepsilon\to\infty} \frac{T_{ex}}{T} = \lim_{\varepsilon\to\infty} \left\{ \frac{\sqrt{1+\frac{3}{4}\,\varepsilon A^2}}{2\pi} \frac{4}{\sqrt{1+\varepsilon A^2}} \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1-k\sin^2 x}} \right\}$$
$$= \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1 - 0{\cdot}5\sin^2 x}} = 0{\cdot}9294.$$

Therefore, for any value of $\varepsilon$, it can be easily proved that the maximal relative error of the period (17) is less than 7%.

## 3. HIGH ORDER APPROXIMATIONS

This section is hinted by an unknown referee. As pointed out by the referee the above procedure cannot give a second order approximation solution because the secular term occurring in the second perturbation equation cannot be eliminated. In order to give a general and versatile procedure, we will apply the Lindstedt–Poincare method [1, 4, 5].

Assume that $\beta^2$ and the solution of equation (5) can be written in the forms

$$\beta^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{19}$$

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots. \tag{20}$$

Substituting equations (19) and (20) into equation (5),

$$(u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots) + (\omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$$
$$+ \varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)^3 + \varepsilon\eta(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) = 0. \tag{21}$$

Collecting coefficients of equal powers of $\varepsilon$, and setting each of the coefficients of $\varepsilon$ equal to zero in equation (21),

$$u_0'' + \omega_0^2 u_0 = 0, \qquad u_1'' + \omega_0^2 u_1 + \omega_1 u_0 + u_0^3 + \eta u_0 = 0, \tag{22, 23}$$

$$u_2'' + \omega_0^2 u_2 + \omega_1 u_1 + \omega_2 u_0 + 3u_0^2 u_1 + \eta u_1 = 0. \tag{24}$$

The initial conditions are replaced by

$$u_0(0) = A, \qquad u_0'(0) = 0, \tag{25}$$

$$\sum_{i=1}^{n} u_i(0) = 0, \qquad \sum_{i=1}^{n} u_i'(0) = 0. \tag{26}$$

The leading term $u_0$ can be readily obtained

$$u_0(t) = A \cos \omega_0 t. \tag{27}$$

The substitution of equation (27) in equation (23) results in

$$\begin{aligned} u_1'' + \omega_0^2 u_1 &= -(\omega_1 + \eta) A \cos \omega_0 t - A^3 \cos^3 \omega_0 t \\ &= -(\omega_1 + \eta + \tfrac{3}{4} A^2) A \cos \omega_0 t - \tfrac{1}{4} A^3 \cos 3\omega_0 t. \end{aligned} \tag{28}$$

To eliminate the secular term needs

$$\omega_1 = -(\eta + \tfrac{3}{4} A^2). \tag{29}$$

Then, we obtain a particular solution of equation (28), which reads

$$u_1 = \frac{A^3}{32\omega_0^2} \cos 3\omega_0 t. \tag{30}$$

Substituting equations (27) and (30) into equation (24) yields

$$\begin{aligned} u_2'' + \omega_0^2 u_2 =& -(\omega_1 + \eta) \frac{A^3}{32\omega_0^2} \cos 3\omega_0 t - \omega_2 A \cos \omega_0 t - \frac{3A^5}{32\omega_0^2} \cos^2 \omega_0 t \cos 3\omega_0 t \\ =& -\left(\omega_1 + \eta + \frac{3A^2}{2}\right) \frac{A^3}{32\omega_0^2} \cos 3\omega_0 t - \left(\omega_2 + \frac{3A^4}{128\omega_0^2}\right) A \cos \omega_0 t \\ &+ \frac{3A^5}{128\omega_0^2} \cos 5\omega_0 t. \end{aligned} \tag{31}$$

Avoiding the presence of a secular term needs

$$\omega_2 = -\frac{3A^4}{128\omega_0^2}. \tag{32}$$

If, for example, its second order approximation is sufficient, then the initial conditions for equation (31) can be expressed as

$$u_2(0) = -\frac{A^3}{32\omega_0^2}, \qquad u_0'(0) = 0. \tag{33}$$

Solving equation (31) with the initial conditions (33), we obtain

$$u_2 = \left(\omega_1 + \eta + \frac{3A^2}{2}\right) \frac{A^3}{256\omega_0^3} \cos 3\omega_0 t + \frac{3A^5}{3072\omega_0^3} \cos 5\omega_0 t + B \cos \omega_0 t, \tag{34}$$

where the constant $B$ can be written in the form

$$B = -\left(\omega_1 + \eta + \frac{3A^2}{2}\right)\frac{A^3}{256\omega_0^3} - \frac{3A^5}{3072\omega_0^3} + \frac{A^3}{32\omega_0^2},$$
$$= -\frac{3A^5}{1024\omega_0^3} - \frac{3A^5}{3072\omega_0^3} + \frac{A^3}{32\omega_0^2},$$
$$= -\frac{A^5}{256\omega_0^3} + \frac{A^3}{32\omega_0^2}. \tag{35}$$

Substituting $\omega_1$ and $\omega_2$ into equation (19), and in view of the identity $\beta^2 + \varepsilon\eta = 1$, we have

$$\omega_0^2 = \beta^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2,$$
$$= \beta^2 + \varepsilon(\eta + \tfrac{3}{4}A^2) + \frac{3\varepsilon^2 A^4}{128\omega_0^2},$$
$$= 1 + \tfrac{3}{4}\varepsilon A^2 + \frac{3\varepsilon^2 A^4}{128\omega_0^2}. \tag{36}$$

The angular frequency $\omega_0$, therefore, can be solved from the above relation (36):

$$\omega_0 = \sqrt{\tfrac{1}{2}(1 + \tfrac{3}{4}\varepsilon A^2) + \tfrac{1}{2}\sqrt{(1 + \tfrac{3}{4}\varepsilon A^2)^2 + \frac{3\varepsilon^2 A^4}{32}}},$$
$$= \sqrt{\tfrac{1}{2}(1 + \tfrac{3}{4}\varepsilon A^2) + \tfrac{1}{2}\sqrt{1 + \tfrac{3}{2}\varepsilon A^2 + \frac{21\varepsilon^2 A^4}{32}}}. \tag{37}$$

We, therefore, obtain the following second order approximate solution:

$$u(t) = A\cos\omega_0 t + \frac{\varepsilon A^3}{32\omega_0^2}\cos 3\omega_0 t$$
$$+ \varepsilon^2\left[\frac{3A^5}{1024\omega_0^3}\cos 3\omega_0 t + \frac{3A^5}{3072\omega_0^3}\cos 5\omega_0 t + B\cos\omega_0 t\right], \tag{38}$$

where $\omega_0$, $\omega_1$, $\omega_2$ and $B$ are defined by equations (37), (29), (32) and (35) respectively.

The period of the solution reads

$$T = \frac{2\pi}{\sqrt{\tfrac{1}{2}(1 + \tfrac{3}{4}\varepsilon A^2) + \tfrac{1}{2}\sqrt{1 + \tfrac{3}{2}\varepsilon A^2 + \tfrac{21}{32}\varepsilon^2 A^4}}}. \tag{39}$$

Observe that for small $\varepsilon$, the present method gives exactly the same results as the standard Lindstedt–Poincare method [4, 5]. And for relatively large values of the parameter, the present method gives almost the same results as the modified Lindstedt–Poincare method [1], from which the following period can be obtained:

$$T = \frac{2\pi}{\sqrt{1/(1-\alpha)(1 - \tfrac{1}{24}\alpha^2 - \tfrac{17}{13824}\alpha^4)}} \quad \text{with } \alpha = \frac{\tfrac{3}{4}\varepsilon A^2}{1 + \tfrac{3}{4}\varepsilon A^2}. \tag{40}$$

To compare with the exact solution, we have

$$\lim_{\varepsilon\to\infty}\frac{T_{ex}}{T}=\lim_{\varepsilon\to\infty}\left\{\frac{\sqrt{\frac{1}{2}(1+\frac{3}{4}\varepsilon A^2)+\frac{1}{2}\sqrt{1+\frac{3}{2}\varepsilon A^2+(21\varepsilon^2A^4/32)}}}{2\pi}\right.$$
$$\left.\frac{4}{\sqrt{1+\varepsilon A^2}}\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{1-k\sin^2 x}}\right\},$$
$$=\frac{2\sqrt{\frac{1}{2}\times\frac{3}{4}+\frac{1}{2}+\sqrt{\frac{21}{32}}}}{\pi}\times 1{\cdot}68575=0{\cdot}9478. \tag{41}$$

Therefore, for any value of $\varepsilon$, the maximal relative error of the period is less than 5·2%.

## 4. COMPARISON WITH THE VARIATIONAL ITERATION METHOD

The second order approximate solution is more accurate than the first order approximate solution obtained by the variational iteration method [6, 7]. The period of Duffing equation obtained by the variational iteration method reads [7]

$$T=\frac{2\pi}{\sqrt{(10+7\varepsilon A^2+\sqrt{(64+104\varepsilon A^2+49\varepsilon^2A^4)}/18}}$$

The maximal relative error of the period is less than 5·7%. Though the accuracy can reach a very high level if its second order approximation can be obtained by the variational iteration method, the procedure is too cumbersome for high order approximations. However, in the present study, we can readily obtain its third or higher order approximate solutions without very cumbersome procedures.

## 5. CONCLUSION

In this paper, we have tentatively presented a kind of new perturbation technique, which does not depend upon the assumption of small parameters. Though the solution of non-linear equations is assumed to have the same form as that of classical perturbation methods, in our study the parameter $\varepsilon$ does not need to be small, and the leading term $u_0$ is obtained from the linearized equation, not only by setting $\varepsilon = 0$. This is the difference between the new technique and old ones. The well-known Duffing equation is illustrated as an example; the results reveal that even its first order approximation has high accuracy; the maximal relative error of the period is less than 7% even when the parameter $\varepsilon\to\infty$.

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